JOURNAL OF APPROXIMATION THEORY 18, 86–98 (1976)

Polynomials of Extremal L_p-Norm on the L_∞-Unit Sphere

D. AMIR

Department of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

AND

Z. ZIEGLER

Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel

Communicated by R. Bojanic

Received February 25, 1975

INTRODUCTION

Several forms of approximation with side conditions have received widespread attention in recent years (see [5] for a review of results along this vein). We discuss in this paper a problem of approximating 0 under a special type of side condition. The approximation we seek is in the L_p-norm, $1 \leqslant p < \infty$, and the approximating polynomials have a fixed L_∞-norm. We give a characterization and properties of such polynomials as well as an estimate for the order of approximation. In a subsequent paper, we investigate the analogous problem with fixed L_q-norm, where q is an arbitrary number greater than p.

The investigation of this type of problem has been initiated by Louboutin [2], who dealt with the L_1-problem and obtained useful partial results. Our work was motivated by Louboutin's paper, and we generalize his results and proceed to answer some of the questions left open in that paper.

In the first section we give a characterization of the polynomials of least L_p-norm in restricted subsets of the L_∞-unit ball, and obtain some properties of such polynomials.

In Section 2 we elaborate the cases $p = 2$ and $p = 1$. In particular, it is established that the zeros of the polynomial of the nth degree of least norm and those of the corresponding polynomial of the $(n + 1)$st degree strictly interlace. This property, for $p = 1$, was conjectured by Louboutin on the basis of numerical evidence.

In Section 3 we derive the exact order of approximation for the case $p = 2$, and deduce estimates for the order of approximation for general p. This is an estimate for the sequence $k_n(p)$, where $k_n(p) = \min[\|f\|_p/\|f\|_\infty ; f \in \pi_n]$.

1. Characterization and General Properties of the Polynomials of least L_p-Norm

Consider the interval [0, 1], and let π_n be the set of polynomials of degree n at most. Let $Q_{n,n}$ be the cone of polynomials of π_n which change sign at most m times in (0, 1). Note that if $n = m$, no restriction is imposed on the polynomials, so that $Q_{n,n} = \pi_n$. Let $Q^*_{n,m}$ be the set of polynomials of $Q_{n,m}$ with $\|f\|_\infty = 1$.

Define, for $1 \leqslant p < \infty$,

$$K_{n,m}(p) = \min\{\|f\|_p ; f \in Q^*_{n,m}\}. \tag{1}$$

We call the polynomials in $Q^*_{n,m}$ whose norm is $K_{n,m}(p)$, "extremal" polynomials. The existence of such polynomials is assured in view of the compactness of $Q^*_{n,m}$. We turn to the discussion of some properties of extremal polynomials.

Lemma 1. *If f^* is extremal in $Q^*_{n,m}$ and $f^*(0) = 1$, then either*

(a) $f^*(1) = \|f^*\|_p$,

or

(b) *f^* has a sign change at* 1.

In particular, if $m = n$, then (a) *prevails.*

Proof. Consider $g_\alpha(t) = f^*(\alpha t)$. Note that for $\alpha \leqslant 1$, g_α is in $Q^*_{n,m}$. Since the function $\|g_\alpha\|_p^p$, as a function of α, has a minimum in [0, 1] for $\alpha = 1$, it follows that

$$(d/d\alpha)[\|g_\alpha\|_p^p]_{\alpha=1} \leqslant 0. \tag{2}$$

A simple computation yields

$$\left[(d/d\alpha)\left\{\int_0^1 |f^*(\alpha t)|^p\, dt\right\}\right]_{\alpha=1} = \left[(d/d\alpha)\left\{(1/\alpha)\int_0^\alpha |f^*(u)|^p\, du\right\}\right]_{\alpha=1}$$
$$= -\|f^*\|_p^p + |f^*(1)|^p.$$

Hence, (2) is equivalent to

$$|f^*(1)| \leqslant \|f^*\|_p. \tag{2'}$$

Since $x^n \in Q^*_{n,m}$ for $m = 0, 1, \ldots, n$, we conclude that

$$K_{m,n}(p) \leqslant \left(\int_0^1 x^{np}\, dx\right)^{1/p} = (np+1)^{-1/p} < 1. \tag{3}$$

Hence the left-hand side of (2′) is strictly smaller than 1, and therefore, by continuity, $\| g_\alpha \|_\infty = 1$ for a right neighborhood of $\alpha = 1$.

If f^* does not change sign at 1, or if $m = n$, then the number of sign changes of g_α for α in a right neighborhood of 1 remains $\leqslant m$, so that $g_\alpha \in Q^*_{n,m}$ for α in such a neighborhood. In this case $\alpha = 1$ is a local two-sided minimum, implying that (2) and (2′) turn into equalities. Q.E.D.

LEMMA 2. *If f^* is extremal in $Q^*_{n,m}$, and $f^*(0) = 1$, then $|f^*(\alpha)| < 1$ for $\alpha \in (0, 1]$.*

Proof. In the course of the proof of Lemma 1, we showed that $|f^*(1)| < 1$. Hence, we restrict our attention to $\alpha \in (0, 1)$. Suppose $|f^*(\alpha)| = 1$, and consider the polynomials $g_\alpha(t) = f^*(\alpha t)$, $h_\alpha(t) = f^*(\alpha + (1-\alpha)t)$. Both polynomials belong to $Q^*_{n,m}$. The minimality of $\| f^* \|_p^p$ implies therefore

$$\begin{aligned} \| f^* \|_p^p &= \alpha \| f^* \|_p^p + (1-\alpha) \| f^* \|_p^p \leqslant \alpha \| g_\alpha \|_p^p + (1-\alpha) \| h_\alpha \|_p^p \\ &= \alpha \int_0^1 |f^*(\alpha t)|^p\, dt + (1-\alpha) \int_0^1 |f^*(\alpha + (1-\alpha)t)|^p\, dt \\ &= \int_0^\alpha |f^*(u)|^p\, du + \int_\alpha^1 |f^*(u)|^p\, du = \| f^* \|_p^p . \end{aligned}$$

Hence the inequality is in fact an equality and g_α is extremal. Noting that $g_\alpha(0) = f^*(0) = 1$, we conclude from the proof of Lemma 1 that $|g_\alpha(1)| < 1$. Since $g_\alpha(1) = f_\alpha^*(\alpha)$, this is in contradiction to our hypothesis. Q.E.D.

LEMMA 3. *If f^* is extremal in $Q^*_{n,m}$, and $f^*(0) = 1$, then*

$$f^{*\prime}(0) = (1/p)[1 - 1/\| f^* \|_p^p]. \tag{4}$$

Proof. Consider the functions

$$g_\alpha(t) = f^*[\alpha + t(1-\alpha)]/f^*(\alpha).$$

For $\alpha \in [0, 1]$ these functions clearly belong to $Q_{n,m}$. Since the values o, $g_\alpha(t)$ for $t \in [0, 1]$ are the values of $f^*(t)/f^*(\alpha)$ for $t \in [\alpha, 1]$, and $f^*(0) = 1$f it follows that $g_\alpha \in Q_{n,m}$ for some left neighborhood of $\alpha = 0$ as well.

Furthermore, since $f^*(t)$ is a polynomial attaining the modulus 1 only at $t = 0$ (by Lemma 2), there exists a right neighborhood of $\alpha = 0$ such that for all α in this neighborhood, $f^*(t)$ attains its maximum modulus on $[\alpha, 1]$ at $t = \alpha$. Hence, for this neighborhood, $\| g_\alpha \|_\infty = 1$ and $g_\alpha \in Q^*_{n,m}$. Thus, the minimum of $\| g_\alpha \|_p$ in this right neighborhood of 0 is attained for $g_0 = f^*$, i.e., for $\alpha = 0$, and we conclude that

$$[(d/d\alpha)[\| g_\alpha \|_p^p]]_{\alpha=0} \geqslant 0. \tag{5}$$

A simple computation, taking into account the positivity of $f^*(\alpha)$ near $\alpha = 0$, yields

$$\begin{aligned}
&\left[\frac{d}{d\alpha}\left\{\int_0^1 \frac{|f^*[\alpha + t(1-\alpha)]|^p}{[f^*(\alpha)]^p}\,dt\right\}\right]_{\alpha=0} \\
&= \left[\frac{d}{d\alpha}\left\{\frac{1}{[f^*(\alpha)]^p}\,\frac{1}{(1-\alpha)}\int_\alpha^1 |f^*(u)|^p\,du\right\}\right]_{\alpha=0} \\
&= \left[\frac{-pf^{*\prime}(\alpha)}{[f^*(\alpha)]^{p+1}}\,\frac{1}{(1-\alpha)}\int_\alpha^1 |f^*(u)|^p\,du\right. \\
&\qquad \left. + \frac{1}{[f^*(\alpha)]^p(1-\alpha)^2}\int_\alpha^1 |f^*(u)|^p\,du - \frac{1}{1-\alpha}\right]_{\alpha=0} \\
&= -pf^{*\prime}(0)\,\|f^*\|_p^p + \|f^*\|_p^p - 1.
\end{aligned}$$

Hence, (5) is equivalent to

$$f^{*\prime}(0) \leqslant (1/p)[1 - \|f^*\|_p^{-p}]. \tag{5$'$}$$

By (3), the right-hand side is strictly negative. Hence f^* is decreasing in a left neighborhood of 0, and therefore for all α in this neighborhood, $f^*(t)$ attains its maximum modulus on $[\alpha, 1]$ at $t = \alpha$. Thus, $\| g_\alpha \|_\infty = 1$ for α in this neighborhood. Hence, $\alpha = 0$ is a local two-sided minimum for $\| g_\alpha \|_p$, so that (5) and (5′) turn into equalities. Q.E.D.

PROPOSITION 4. *If f^* is extremal in $Q^*_{n,m}$, then*

$$|f^*(\alpha)| < 1 \qquad \textit{for} \quad \alpha \in (0, 1).$$

Proof. Assuming that $f^*(\alpha) = 1$, we define $h_\alpha(t) = f^*[\alpha + t(1 - \alpha)]$. As in the proof of Lemma 2, we conclude that $h_\alpha(t)$ is extremal. Observing that $h_\alpha(0) = f^*(\alpha) = 1$, we now apply Lemma 3 and deduce that $h_\alpha'(0) < 0$. However, $h_\alpha'(0) = (1 - \alpha) f^{*\prime}(\alpha) = 0$, since α is an interior maximum of f^*. Hence, there exists a contradiction. Q.E.D.

We have thus shown that if f^* is extremal, we may assume that $f^*(0) = 1$. The other extremal functions are obtainable by reflection with respect to $y = 0$ or $x = \frac{1}{2}$. *Henceforth, an extremal f will be assumed to have the form* $1 + \sum_1^n c_j x^j$.

LEMMA 5. *If f^* is extremal in $Q_{n,m}^*$ and $f^*(0) = 1$, then f^* has n zeros, counting multiplicities, in* $(0, 1]$.

Proof. Assume that $f^*(x) = s(x)\, e(x)$, where $s(x)$ satisfies $s(x) \geqslant c > 0$ in $[0, 1]$ and $\deg s(x) \geqslant 1$. Then $f_1(x) = e(x)[s(x) - cx/2]$ satisfies $\| f_1 \|_\infty = f_1(0) = 1$, so that $f_1 \in Q_{n,m}^*$, while clearly $\| f_1 \|_p < \| f^* \|_p$, contradicting the extremality. Q.E.D.

PROPOSITION 6. *Let $f \in Q_{n,m}^*$ with $f(0) = 1$ possess the decomposition $f(x) = s(x)\, p(x)$, where $s(0) = 1$, $s(x) \geqslant 0$ on $[0, 1]$, $\deg s(x) = k$, $p(1) \neq 0$, and $p(x)$ has no double roots in $(0, 1)$. Then f is minimal in $Q_{n,m}^*$ among all polynomials of the form $s(x)\, q(x)$ if and only if it satisfies the orthogonality conditions*

$$\int_0^1 s(t) \mid f(t)|^{p-1} \cdot \operatorname{sgn} f(t) \cdot t^j \, dt = 0, \qquad j = 1,\ldots, n-k. \tag{6}$$

Proof. (a) Assume f is extremal and let $p(t) = 1 + \sum_1^{n-k} v_i t^i$. Define

$$h_u = h(\bar{u}; t) = s(t) \left[1 + \sum_{i=1}^{n-k} u_i t^i \right].$$

Since f is extremal, $\bar{v}$ is a minimum for $\| h_u \|_p$. Since $h_v'(0)$ is strictly negative and $p(t)$ has no double zeros in $(0, 1)$, it follows that a small perturbation of the v_j's leaves h_u in $Q_{n,m}^*$, with $\| h_u \|_\infty = h_u(0) = 1$.

Thus, we have

$$0 = \left[(\partial/\partial u_j) \int_0^1 | h_u(t)|^p \, dt \right]_{\bar{u}=\bar{v}} = \left[(\partial/\partial u_j) \int_0^1 [s(t)]^p \left| 1 + \sum_1^{n-k} u_i t^i \right|^p dt \right]_{\bar{u}=\bar{v}}$$

$$= p \left[\int_0^1 [s(t)]^p \left| 1 + \sum_1^{n-k} u_i t^i \right|^{p-1} \operatorname{sgn} \left(1 + \sum_1^{n-k} u_i t^i \right) \cdot t^j \, dt \right]_{\bar{u}=\bar{v}}$$

$$= p \int_0^1 s(t) \mid f(t)|^{p-1} \operatorname{sgn} f(t) \cdot t^j \, dt, \qquad j = 1,\ldots, n-k,$$

establishing (6).

(b) Assume $f(t)$ satisfies (6). Let $f_1(t) = s(t)\, q(t)$. Then $q(t) - p(t)$ is a polynomial of degree $n - k$ with no constant term. Observe now that

$$\begin{aligned}\int_0^1 |f(t)|^p\,dt &= \int_0^1 |f(t)|^{p-1}\cdot f(t)\operatorname{sgn} f(t)\,dt\\ &= \int_0^1 |f(t)|^{p-1} s(t)\,p(t)\operatorname{sgn} f(t)\,dt \qquad \text{(using (6) here)}\\ &= \int_0^1 |f(t)|^{p-1} s(t)\,q(t)\operatorname{sgn} f(t)\,dt\\ &\leqslant \int_0^1 |f(t)|^{p-1}\,|f_1(t)|\,dt \qquad \text{(Hölder's inequality)}\\ &\leqslant \Big[\int_0^1 |f(t)|^{(p-1)q}\,dt\Big]^{1/q}\Big(\int_0^1 |f_1(t)|\,dt\Big)^{1/p}.\end{aligned}$$

Noting that $(p-1)q = p$, and $1-(1/q) = 1/p$, we conclude that

$$\Big(\int_0^1 |f(t)|^p\,dt\Big)^{1/p} \leqslant \Big(\int_0^1 |f_1(t)|^p\,dt\Big)^{1/p}. \qquad \text{Q.E.D.}$$

PROPOSITION 7. *Let f^* be extremal in $Q^*_{n,m}$, and let $f^*(0) = 1$. Assume that $f^*(x) = s(x)\,p(x)$, where $s(0) = 1$, $s(x) \geqslant 0$ on $[0, 1]$, $p(1) \neq 0$, and $p(x)$ has no multiple roots in $(0, 1)$. Then*

(1) $\deg p(x) = m, \qquad \deg s(x) = n - m$;

(2) *$f(x)$ satisfies the orthogonality relations* (6) *with $n - k = m$*;

(3) *$f(x)$ has exactly m sign changes in* $(0, 1)$.

Proof. If $\deg s = k$, then, by Proposition 6, $f^*(x)$ satisfies (6). We show now that $f^*(x)$ has exactly $n-k$ sign changes in $(0, 1)$.

Indeed, let $0 < t_1 < \cdots < t_r < 1$ be the points where f^* changes sign in $(0, 1)$. Since $s(t) \geqslant 0$ and $\deg p = n - k$, we conclude that $r \leqslant n-k$. Suppose now that $r < n-k$, and choose $w(t) = ts(t)\prod_{i=1}^{r}(t_i - t)$. Then

$$w(t) = s(t)\sum_1^{n-k} a_j t^j \qquad \text{and} \qquad \operatorname{sgn} f^*(t) = \operatorname{sgn} w(t) \text{ in } (0, 1).$$

Hence

$$\begin{aligned}0 &< \int_0^1 |f(t)|^{p-1}\,w(t)\operatorname{sgn} w(t)\,dt\\ &= \int_0^1 |f(t)|^{p-1} s(t)\sum_1^{n-k} a_j t^j\cdot\operatorname{sgn} f(t)\,dt\\ &= \sum_1^{n-k} a_j\int_0^1 |f(t)|^{p-1} s(t)\operatorname{sgn} f^*(t)\cdot t^j\,dt = 0.\end{aligned}$$

The contradiction establishes that r must be equal to $n - k$.

Since f^* belongs to $Q^*_{n,m}$, we conclude that $n - k \leqslant m$. Assume now that $k > n - m$, and let $s(x) = s_1(x)\, q(x)$ where $\deg s_1 = n - m$, $\deg q \geqslant 1$, $q(0) = 1$, and $q(x) \geqslant 0$ on $(0, 1)$. We have $f^*(x) = s_1(x) \cdot [q(x)\, p(x)]$. We repeat now verbatim the perturbation argument used in the proof of Proposition 6, noting that the perturbation leaves us in $Q^*_{n,m}$ since $\deg[qp] = m$. Thus we deduce that f^* satisfies m orthogonality conditions of the form (6). Hence, by a previously used argument, f^* has exactly m sign changes, in contradiction to the propetries of $q(x)$. Q.E.D.

COROLLARY 8. *Let f^* be extremal in $Q^*_{n,m}$ and $f^*(0) = 1$. If $n - m$ is even then $|f^*(1)| = \|f^*\|_p$, while if $n - m$ is odd then $f^*(1) = 0$.*

Proof. Using Lemma 1 and the fact that all zeros of f^* are in $(0, 1]$, the corollary follows from the fact that $\deg s = n - m$. Q.E.D.

COROLLARY 9. *The solution to the extremal problem for $Q^*_{n,n}$ and the L_p-norm, $1 \leqslant p < \infty$, is unique up to reflection.*

Proof. The set

$$A_n = \{f; f \in Q^*_{n,n}, f(0) = 1\}$$

is convex, so that uniqueness for $1 < p < \infty$ is assured by virtue of the strict convexity of the norm.

Assume now that $p = 1$ and let f and g be two polynomials of A_n of least L_1-norm. Then

$$\|(f + g)/2\|_1 = \|f\|_1 = \|g\|_1 .$$

Hence, $\|f + g\|_1 = \|f\|_1 + \|g\|_1$. Since f and g are polynomials, this equality implies that f and g agree in sign everywhere on $(0, 1)$.

Using Proposition 7 for $m = n$, we conclude that f and g have the same n points of sign change, i.e., they share the same n zeros. Since both are polynomials of degree n and are equal at $t = 0$, they must be identical. Q.E.D.

Notation. The unique extremal polynomial in $Q^*_{n,n}$ for the L_p-norm possessing the value 1 for $t = 0$ will be denoted by $V_{n,p}(x)$.

2. THE SPECIAL CASES $p = 2$ AND $p = 1$

In this section we treat in more detail the minimization in $Q^*_{n,n}$ for $p = 2$ and $p = 1$. The extremal polynomial $V_{n,2}$ is explicitly identified as a Jacobi polynomial. We conclude that the zeros of $V_{n,2}$ and $V_{n+1,2}$ interlace. An analogous property is then established for $V_{n,1}$ and $V_{n+1,1}$ through a refined analysis.

PROPOSITION 10. *The extremal polynomials $V_{n,2}(x)$ are orthogonal on* (0, 1) *with respect to the weight function t. Hence they can be identified as the Jacobi polynomials $(-1)^n P_n^{(0,1)}(2x-1)/(n+1)$; the zeros of $V_{n,2}$ and those of $V_{n+1,2}$ interlace.*

Proof. Using Proposition 7 for $n = m$, we note that $\deg s = k = 0$, so that $V_{n,2}$ satisfies the orthogonality conditions

$$\int_0^1 |V_{n,2}(t)| \cdot \operatorname{sgn} V_{n,2}(t) \cdot t^j \, dt = 0, \qquad j = 1,..., n,$$

or

$$\int_0^1 V_{n,2}(t) \cdot t^i(t \, dt) = 0, \qquad i = 0, 1,..., n-1. \tag{7}$$

Hence, $\{V_{n,2}\}$ are the orthogonal polynomials with respect to $w(t) = t$ on [0, 1]. They are therefore constant multiples of the Jacobi polynomials $P_n^{(0,1)}(2x-1)$ (see [4, p. 58]). Using our normalization we have

$$1 = V_{n,2}(0) = c_n P_n^{(0,1)}(-1) = (-1)^n \binom{n+1}{n} = (-1)^n(n+1),$$

so that $V_{n,2}(x) = (-1)^n P_n^{(0,1)}(2x-1)/(n+1)$.

The interlacing properties are a consequence of the general theorem for orthogonal polynomials (see [4, p. 46]). Q.E.D.

We turn next to the characterization of $V_{n,1}(x)$. Using Proposition 7, we see that $V_{n,1}(x)$ satisfies

$$\int_0^1 \operatorname{sgn} V_{n,1}(t) \cdot t^j \, dt, \qquad j = 1,..., n. \tag{8}$$

Condition (8) was discovered by Louboutin [2], who proceeded to solve the resulting system of equations numerically for $n \leqslant 14$, and on the basis of the results conjectured that the roots of $V_{n,1}$ and those of $V_{n+1,1}$ interlace. We shall now prove that is indeed true, giving support to G. Glaeser's point of view (expressed in the introduction to the collection [2]) that experimenting with a computer may be a good technique to generate new theorems.

PROPOSITION 11. *The extremal polynomials $V_{n,1}(t)$ can be written as*

$$V_{n,1}(t) = \prod_{i=1}^{n} (1 - t/t_i),$$

where $(t_1, \ldots, t_n)$ *is the unique solution of the system*

$$2 \sum_{i=1}^{n} (-1)^{i-1} t_i^k + (-1)^n = 0, \quad k = 2, \ldots, n+1, \quad 0 \leqslant t_i \leqslant t_{i+1} \leqslant 1. \tag{9}$$

The points $\{t_i\}_1^n$ *are distinct; the points corresponding to* $V_{n,1}$ *and to* $V_{n+1,1}$ *interlace.*

Furthermore, we have

$$K_{n,n}(1) = 2 \sum_{1}^{n} (-1)^{i-1} t_i + (-1)^n. \tag{10}$$

Proof. System (8) is equivalent to

$$\sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} (-1)^i t^j \, dt = 0, \qquad j = 1, \ldots, n,$$

where $V_{n,1}$ changes sign at each t_i, $i = 1, \ldots, n$, and $t_0 = 0$, $t_{n+1} = 1$. After integrating, this system becomes

$$\sum_{i=0}^{n} (-1)^i (t_{i+1}^k - t_i^k) = 0, \qquad k = 1, \ldots, n,$$

which is equivalent to (9). Hence, $V_{n,1}$ has the form stipulated in the theorem, and, by Proposition 7 for $m = n$, there are n distinct roots satisfying (9).

For the uniqueness and interlacing properties we apply induction. Let $t_{n,i}$ $(i = 1, \ldots, n)$ be the roots of $V_{n,1}$. It is immediate to verify that system (9) has a unique solution for $n = 1$ and $n = 2$ and that $t_{2,1} < t_{1,1} < t_{2,2}$. Suppose we have proved these properties up to $n = m$, and consider the system (9) for $n = m + 1$. The vectors $(0, t_{m,1}, t_{m,2}, \ldots, t_{m,m})$ and $(t_{m,1}, \ldots, t_{m,m}, 1)$ solve the first m equations ($k = 2, \ldots, m + 1$). We want to change the parameter t monotonically from 1 to t_{mm} and consider the system of equations

$$2 \sum_{i=1}^{m} (-1)^{i-1} v_i^k + 2(-1)^m t^k + (-1)^{m+1} = 0, \qquad k = 2, \ldots, m+1, \tag{11}$$
$$0 \leqslant v_i \leqslant v_{i+1} \leqslant 1.$$

Rewriting in a vectorial form, we have $\bar{f}(\bar{v}, t) = 0$, where $f(\bar{v}, t)_k = 2 \sum_{i=1}^{m} (-1)^{i-1} v_i^{k+1} + 2(-1)^m t^{k+1} + (-1)^{m+1}$. The Jacobian $\partial \bar{f} / \partial \bar{v}$ is $(-1)^{m(m-1)/2} 2^m (m+1)! \prod_{i=1}^{m} v_i \prod_{i<j} (v_j - v_i)$, so that it does not vanish, provided that $v_{i+1} > v_i > 0$. When this happens, the (local) solution $\bar{v} = \bar{v}(t)$ has the derivatives $dv_j/dt = (t/v_j) \prod_{i<j} ((t - v_i)/(v_j - v_i)) \prod_{i>j} ((t - v_i)/(v_i - v_j))$ which are positive if $t > v_{i+1} > v_i > 0$, for all i.

Let t_0 be the infimal such that, for every $t \in (t_0, 1]$, system (11) has a unique solution satisfying $0 < v_i(t) < v_{i+1}(t) < t$. Since for $t = 1$ system (11) reduces to system (9) for $n = m$ by the induction hypothesis it has the unique solution $v_i(t) = t_{m,i}$. By the implicit function theorem, there is a neighborhood G of 1 and a neighborhood W of $(t_{m,1},..., t_{m,m})$ (which we may assume to separate $t_{m,i}$ and $t_{m,i+1}$ for all i, and $t_{m,m}$ and 1) such that for all $t \in G$ there exists a unique solution $v(t)$ in W. If $\bar{v}'(t)$ is another solution for such t, and $t \to 1$, then the Bolzano-Weierstrass theorem implies the existence of a second solution $\bar{v}'(1)$, so that we may assume that in G the solution of (11) is unique. By the implicit function theorem, it depends continuously (and even smoothly) on t (see [1, Theorem 6.74, p. 248]).

This shows that $t_0 < 1$. By the definition of t_0, $v_j(t)$ decreases monotonically with t, so that $v_j(t_0) = \lim_{t \to t_0} v_j(t)$ exists and by continuity this solves system (11) for $t = t_0$. We claim that $v_1(t_0) = 0$. Suppose $v_1(t_0) \neq 0$. We cannot have $v_i(t_0) = v_{i+1}(t_0)$, since in this case $(v_1(t_0),..., v_{i-1}(t_0), v_{i+2}(t_0),..., v_m(t_0), t_0)$ is a solution of system (9) for $n = m - 1$, and, by the induction hypothesis, $v_1(t_0) = t_{m-1,1} > t_{m,1}$, while $v_1(1) = t_{m,1}$, which cannot happen since $v_1(t_0)$ decreases from 1 to t_0. Similarly, we cannot have $v_m(t_0) = t_0$. If the solution $\bar{v}(t_0)$ is not unique, a second solution $\bar{v}'(t_0)$ is subject to the same restrictions and, in particular, has a nonzero Jacobian, which will yield a second solution also in a neighborhood of t_0, contradicting the definition of t_0.

This shows, analogously to the discussion about 1, that t_0 is not the infimum. Therefore $v_1(t_0) = 0$. In this case $(v_2(t_0),..., v_m(t_0), 0)$ is a solution of system (9) at t_0, and, by the induction hypothesis, $v_i(t_0) = t_{m,i-1}$. But considering again the last equation

$$2 \sum_{i=1}^{m+1} (-1)^{i-1} v_i^{m+2} + (-1)^{m+1} = 0,$$

we see that the left-hand side changes sign (continuously) when $t_{m+1} = t$ changes from 1 to $t_{m,m}$ and $t_i = v_i(t)$ for $i = 1,..., m$. Hence, there is a unique solution of system (9) for $n = m + 1$ and the monotone dependence of the $v_i(t)$ on t yields the interlacing property for $m + 1$ and m.

In order to prove (10) we note that $V_{n,1}(t) = 1 + \sum_1^n c_i t^i$ and make use of the orthogonality conditions (8), to obtain

$$K_{n,n}(1) = \| V_{n,1} \|_1 = \int_0^1 V_{n\,1}^{(t)} \operatorname{sgn} V_{n,1}(t)\, dt = \int_0^1 \operatorname{sgn} V_{n,1}(t)\, dt$$

$$= \sum_{i=0}^{n} (-1)^i (t_{i+1} - t_i).$$

This reduces to (10) after a simple rearrangement. Q.E.D.

3. The Degree of Approximation

In this section we present some precise information concerning the degree of approximation. We will then give estimates, for general p, of $k_n(p) = K_{n,n}(p) = \min\{\|f\|_p/\|f\|_\infty ; f \in \pi_n\}$.

Proposition 12. $k_n(2) = (n+1)^{-1}$.

Proof. We appeal to Corollary 8 and Proposition 10, deducing that

$$\begin{aligned} k_n(2) = K_{n,n}(2) &= \| V_{n,2} \|_2 = | V_{n,2}(1)| \\ &= (1/(n+1)) \, | P_n^{(0,1)}(1)| = 1/(n+1), \end{aligned}$$

where the last equality was taken from [4, p. 58]. Q.E.D.

Corollary 13. $K_{2n,0}(1) = (n+1)^{-2}$.

Proof. Lemma 5 implies that if f^* is the extremal in $Q_{2n,0}^*$ then it is the square of a polynomial $g \in Q_{n,n}^*$. Trivially, a square of a polynomial in $Q_{n,n}^*$ belongs to $Q_{2n,0}^*$. Hence

$$K_{2n,0}(1) = \min_{g \in Q_{n,n}^*} \int_0^1 g^2(t) \, dt = [K_{n,n}(2)]^2.$$

The corollary now is a direct consequence of Proposition 12. Q.E.D.

Proposition 14. $K_{2n-1,0}(1) = 1/n(n+1)$.

Proof. By Lemma 5 and Corollary 8 an extremal function must have the form $(1-t)\, g^2(t)$, where $g \in Q_{n-1,n-1}^*$. Hence,

$$K_{2n-1,0}(1) = \min_{g \in Q_{n-1,n-1}^*} \int_0^1 g^2(t)(1-t) \, dt.$$

Thus, it suffices to show that the minimal norm of a polynomial of the hyperplane $A_{n-1} = \{f; f \in Q_{n-1,n-1}^*, f(0) = 1\}$ in the inner product space $Q_{n-1,n-1}$ with the weight function $(1-t)$ is equal to $[n(n+1)]^{-1/2}$.

Consider the orthogonal family corresponding to this weight function. This is the normalized family $J_k(t)$ corresponding to the Jacobi polynomials $P_k^{(0,1)}(2x-1)$, $k = 0, 1, \ldots, n-1$. They satisfy the Rodrigues formula [4, p. 67–68]

$$(1-t)\, J_k(t) = \frac{(2(k+1))^{1/2}}{k!} \frac{d^k}{dt^k} [t^k (1-t)^{k+1}], \qquad k = 0, 1, \ldots, n-1.$$

Hence, we conclude that $J_k(0) = (2(k+1))^{1/2}$, $k = 0, 1, \ldots, n-1$, $A_{n-1} = \{\bar{a}; \bar{a} = (a_0, \ldots, a_{n-1}), \sum_{k=0}^{n-1} (2(k+1))^{1/2} a_k = 1\}$. Thus, the direction orthogonal to the hyperplane is given by

$$g_0(t) = \sum_{k=0}^{n-1} (2(k+1))^{1/2} J_k.$$

Note that $g_0(0) = \sum_0^{n-1} (2(k+1))^{1/2} \cdot (2(k+1))^{1/2} = 2 \sum_0^{n-1} (k+1) = n(n+1)$. Hence, the minimal norm for a function of $Q^*_{n-1,n-1}$ is given by

$$\begin{aligned}
&\left[\int_0^1 [g_0(t)/g_0(0)]^2 (1-t)\, dt\right]^{1/2} \\
&\quad = (1/n(n+1)) \left[\int_0^1 \left[\sum_0^{n-1} (2(k+1))^{1/2} J_k(t)\right]^2 (1-t)\, dt\right]^{1/2} \\
&\quad = (1/n(n+1)) \left[\sum_0^{n-1} 2(k+1)\right]^{1/2} = [n(n+1)]^{-1/2}. \qquad \text{Q.E.D.}
\end{aligned}$$

Remark. The results in Propositions 13 and 14 represent a substantial improvement of the results of B. Sendov as quoted by Mitrinovic [3, p. 230]. He obtains there

$$K_{2n,0}(1) \leqslant 1/(2n+1), \qquad K_{2n-1,0} \leqslant 1/2n.$$

COROLLARY 15. *We have*

$$\begin{aligned}
1/(n+1)^2 \leqslant k_n(1) &\leqslant 4/(n+2)^2, && n \text{ even}, \\
&\leqslant 4/(n+1)(n+3), && n \text{ odd}.
\end{aligned}$$

Proof. Let $f \in Q^*_{n,n}$. Then $|f(x)|^2 \leqslant |f(x)|$, for all x. Hence we obtain, using Proposition 12, that

$$\begin{aligned}
k_n(1) = K_{n,n}(1) &= \min_{f \in Q^*_{n,n}} \int |f(x)|\, dx \\
&\geqslant \min_{f \in Q^*_{n,n}} \int |f(x)|^2\, dx = [K_{n,n}(2)]^2 = 1/(n+1)^2. \qquad (12)
\end{aligned}$$

On the other hand, since $Q^*_{n,0} \subset Q^*_{n,n}$, we obtain from Propositions 13 and 14 that

$$\begin{aligned}
k_n(1) &\leqslant \min_{f \in Q^*_{n,0}} \int |f(x)|\, dx \\
&= K_{n,0}(1) = \frac{1}{(n/2+1)^2}, && \text{if } n \text{ is even}, \\
&= \frac{1}{((n+1)/2) \cdot (((n+1)/2)+1)}, && \text{if } n \text{ is odd}.
\end{aligned}$$

Q.E.D.

On the basis of the numerical evidence in [2] it seems likely that the true value is close to the bound from above in the last corollary.

Similar considerations yield

COROLLARY 16. *We have, for all* $p > 1$,

$$1/(n+1)^2 \leqslant k_n(p) \leqslant [4/(n+2)^2]^{1/p}, \qquad \textit{for } n \textit{ even},$$
$$\leqslant [4/(n+1)(n+3)]^{1/p}, \qquad \textit{for } n \textit{ odd}.$$

For $p > 2$ *the upper bound can be improved to* $(n+1)^{-2/p}$ *for all* n, *and the lower bound to* $1/(n+1)$, *while for* $p < 2$ *the lower bound can be improved to* $(n+1)^{-2/p}$.

Remarks. (1) The question of relating two norms of a polynomial is related to the vectorial approximation problem. In particular, see [6] where the curve of $\| f - p \|_2$ vs $\| f - p \|_\infty$ is examined and profitably utilized. Our results throw some light on the asymptotical shape of such a curve in special cases.

(2) The extremal polynomial for $p = 2$ was discovered by different methods and given in a different representation, by Szego [4, p. 180]. We wish to thank Professor R. A. Askey for drawing our attention to this fact.

ACKNOWLEDGMENT

We wish to thank Dr. E. Lapidot for some helpful conversations.

REFERENCES

1. P. C. CURTIS, JR., "Multivariate Calculus," Wiley, New York, 1972.
2. R. LOUBOUTIN, Inégalité entre les normes L^1 et L^∞ sur l'espace des polynômes, *in* "Le Prolongateur de Whitney—Tome II" (G. Glaeser, Ed.), pp. E1–E11, Rennes, 1967.
3. D. S. MITRINOVIČ, "Analytic Inequalities," Springer-Verlag, Berlin, 1970.
4. G. SZEGÖ, "Orthogonal Polynomials," Vol. 23, Rev. ed., Amer. Math. Soc. Colloquium New York, 1959.
5. G. D. TAYLOR, Uniform approximation with side conditions, *in* "Approximation Theory" (G. G. Lorentz, Ed.), pp. 495–503, Academic Press, New York, 1973.
6. A. BACOPOULOS AND A. DUBUQUE, An efficient blend of the first algorithm of Remes with quadratic programming, to appear.

Printed in Belgium